# Some qualitative features of homogeneous stability and Turing instabilit two-species competition in a batch fermenter*

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A model of two species of microorganisms competing in a spatially heterogeneous batch fermente is analysed. Using stability conditions without computation, practically useful inferences are derive about the effects of important parameters such as the maximum specific growth rates and the inoculum sizes. A threshold condition is derived for Turing instability, and it is argued that a parametrically sen tive population is more likely to 'jump' from a stable state to an unstable one, leading to spontaneou pattern formation.

Classical reaction-diffusion theory teaches that diffusion has a stabilising effect on a (biological or chemical) reaction. Turing's seminal work[1] showed that, under certain conditions, the presence of diffusion can introduce instability, leading to spatial patterns. Since then, a variety of systems have been studied and there have been a number of developments[2].

Turing instability has been shown to occur in systems as diverse as heterogenous catalysis, the skins of animals, hallucination in vision, and interactions of microbial populations. In this brief study, we consider an important and instructive system of two microbial species competing for the same nutrient in a batch heterogeneous environment. Effluent treatment in a lagoon, biogas generation from animal faeces and the production of yogurt are some common examples. In the fermentative production of yogurt, *Lactobacillus bulgarius* and *Streptococcus thermophilus* compete for the nutrient lactose and both produce lactic acid as an inhibitor[3]. Recently Shama[4] has demonstrated that a co-culture of *Bacillus stearothermophilus* and *Clostridium thermocellum* produces more ethanol and greater conversion of reducing sugars than *Bacillus stearothermophilus* alone.

Using well-established models of microbial interaction and diffusion, we show that simple qualitative analyses can provide insight into the behaviour of the species. We derive inferences about the effects of parameters on the stability of the homogeneous (spatially uniform) system and the onset of instability through diffusion. It is also suggested that, as in chemical reactions, environmental fluc-

tuations which cause small changes in the para ters can destabilise a sensitive population wh deterministically stable even with diffusion.

## Theory

In large fermenters, specially those cont non-Newtonian broths, it is well known tha romixing is imperfect and diffusional effe be significant. The dynamics of two microo isms competing for the same resource in su nonhomogeneous environment without flow scribed by two differential equations.

$$\frac{\partial u_1}{\partial \tau} = \gamma f_1(u_1, u_2) + \frac{\partial^2 u_1}{\partial \zeta^2}$$

$$\frac{\partial u_2}{\partial \tau} = \gamma f_2(u_1, u_2) + d \frac{\partial^2 u_2}{\partial \zeta^2}$$

To the right of the equalities, $f_1$ and $f_2$ are growth (or reaction) terms and the partial der tives represent Fickian diffusion of the specie and $d$ are parameters whose significance wi shown.

In the absence of diffusion the homogen state is a simpler version of Eqs (1) and (2).

$$\frac{d u_1}{d \tau} = \gamma f_1(u_1, u_2)$$

$$\frac{d u_2}{d \tau} = \gamma f_2(u_1, u_2)$$

Eqs (1) and (2) are subject to the boun conditions

$$\frac{\partial u_1}{\partial \zeta} = \frac{\partial u_2}{\partial \zeta} \text{ at } \zeta = 0,1$$

and the initial condition

$$u_1 = u_2 = 0 \text{ at } \tau = 0 \qquad \ldots (6)$$

(To account for the biological requirement of finite initial concentrations of the microorganisms, $u_1$ and $u_2$ may be considered, without loss of generality, to be deviations from these starting concentrations).

Linear stability of the homogeneous steady state is assured by requiring that $Re(\lambda_1, \lambda_2) < 0$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the community matrix of Eqs (3) and (4). This requirement is met by specifying two conditions[5].

$$tr A = \frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} < 0 \qquad \ldots (7)$$

$$|A| = \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} > 0 \qquad \ldots (8)$$

where $A$ is the community matrix.

For spatial diffusion to cause (Turing) instability, Eqs (7), (8) and two others given below constitute necessary and sufficient conditions[5].

$$d \frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} > 0 \qquad \ldots (9)$$

$$\left( d \frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} \right)^2 - 4d \left( \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} \right) > 0 \qquad \ldots (10)$$

For the growth of two competing species, the functions $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ can be specified according to the Lotka-Volterra model. [A recent application has been to a co-culture of *Streptococcus thermophilus* and *Lactobacillus bulgaricus*[3]. Then Eqs (1) and (2) can be written as:

$$\frac{\partial u_1}{\partial \tau} = \gamma u_1 \left[ 1 - \frac{\alpha_1}{\mu_1} (\beta_1 x_{10} u_1 + \beta_2 x_{20} u_2) \right] + \frac{\partial^2 u_1}{\partial \zeta^2} \qquad \ldots (11)$$

$$\frac{\partial u_2}{\partial \tau} = \gamma u_2 \left[ \frac{\mu_2}{\mu_1} - \frac{\alpha_2}{\mu_1} (\beta_1 x_{20} u_1 + \beta_2 x_{20} u_2) \right] + d \frac{\partial^2 u_2}{\partial \zeta^2} \qquad \ldots (12)$$

Here $\tau = D_1 t / L^2$; $\zeta = z/L$; $\gamma = \mu_1 L^2/d_1$; $u_1 = x_1/x_{10}$; $u_2 = x_2/x_{20}$; and $d = D_2/D_1 \cdot \alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are positive constants. The specific growth rates $\mu_1$ and $\mu_2$ are with respect to the nutrient for which the species compete.

The logistic equation, which is often used to simulate competitive microbial growth[6,7], is a special case where $\beta_1 = \beta_2$.

## The Homogeneous Steady State

We begin by analysing the linear stability of the homogeneous steady state. Eqs (11) and (12) are written in the form of Eqs (3) and (4), and then condition (7) is applied to obtain

$$(2/\mu_1)(\alpha_1 x_{10} u_1 + \alpha_2 \beta_2 x_{20} u_2) > 1 + \mu_2/\mu_1 \qquad \ldots (13)$$

Condition (8) results in Eq. (14).

$$\beta_1 x_{10} u_1 [h(u_1, u_2) - (\alpha_2/\mu_1) - 2(\alpha_1/\mu_1)(\mu_2/\mu_1)]$$
$$+ \beta_2 x_{20} u_2 [h(u_1, u_2) - 2(\alpha_2/\mu_1) - (\alpha_1/\mu_1)(\mu_2/\mu_1)]$$
$$+ (\mu_2/\mu_1) + (\alpha_1 \alpha_2/\mu_1^2)(\beta_1 x_{10} u_1 + \beta_2 x_{20} u_2)^2 > 0 \qquad \ldots (14)$$

where

$$h(u_1, u_2) = u_1 \beta_1 x_{10} \alpha_1 \alpha_2/\mu_1^2 + u_2 \beta_2 x_{20} \alpha_1 \alpha_2/\mu_1^2.$$

The inequality is satisfied if the terms in square brackets are non-negative. Combining the resulting two inequalities then leads to:

$$\frac{u_1 \beta_1 x_{10} + u_2 \beta_2 x_{20}}{\mu_1} > sup. \left[ \left( \frac{1}{\alpha_1} + \frac{2\mu_2}{\alpha_2 \mu_1} \right), \left( \frac{2}{\alpha_1} + \frac{\mu_2}{\alpha_2 \mu_1} \right) \right] \qquad \ldots (15)$$

Eqs (13) and (15) suffice to guarantee linear homogeneous stability. Notice that if $1 < \alpha_1$, $\alpha_2 < 2$, Eq. (13) implies Eq. (15). In general six parameters ($\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\mu_1$, $\mu_2$) and two initial conditions ($x_{10}$ and $x_{20}$) have to be specified to define the stability region(s) in the $u_1 - u_2$ plane. To derive qualitative inferences about the behaviour of the system, we consider two cases.

*Case (i)*:

$$\frac{1}{\alpha_1} + \frac{2\mu_2}{\alpha_2 \mu_1} = 1 + \frac{\mu_2}{\mu_1} \qquad \ldots (16)$$

*Case (ii)*:

$$\frac{2}{\alpha_1} + \frac{\mu_2}{\alpha_2 \mu_1} = 1 + \frac{\mu_2}{\mu_1} \qquad \ldots (17)$$

Apart from reducing the dimensionality of the problem, these equalities specify a relationship between $\alpha_1$ and $\alpha_2$ parameterised by $\mu_2/\mu_1$, the ratio of the maximum specific growth rates of the microorganisms. The physical implication of this ratio is analysed below.

Figs 1 and 2 depict Eqs (16) and (17) for three values of $\mu_2/\mu_1$. We recognise first that both the sets of curves attain asymptotic values as either
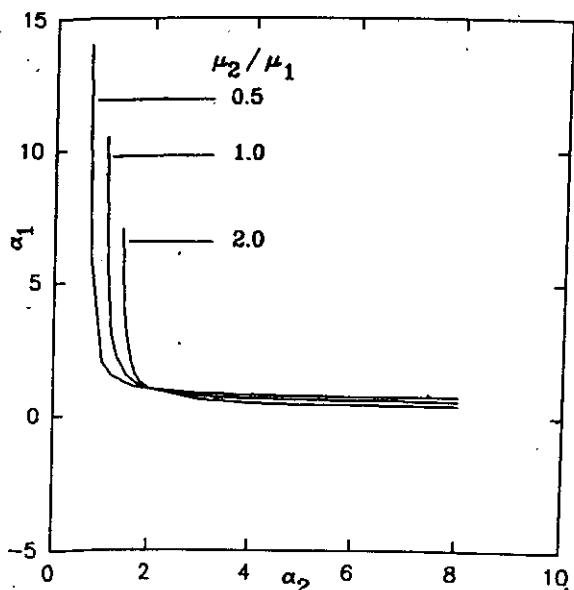
Fig. 1—Effect of maximum specific growth rates on the $\alpha_1$-$\alpha_2$ profile for case (i) [Eq. (16)]
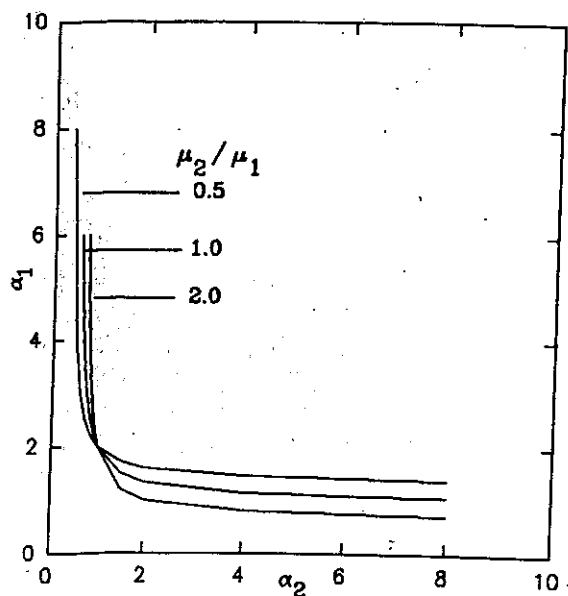


Fig. 2—Effect of maximum specific growth rates on the $\alpha_1$-$\alpha_2$ profile for case (ii) [Eq. (17)]

variable goes beyond about 6. Both the effect of $\mu_2/\mu_1$ and the variation of $\alpha_1$ with $\alpha_2$, for a given value of $\mu_2/\mu_1$, are strongest for moderately small values of $\alpha_1$ and $\alpha_2$. These observations imply that competition has significance when both species are comparable in their metabolic and growth rates. When either species overwhelms the other (in growth-related functions such as resource utilisation or resistance to inhibitors), the parametric effect on the weaker species is negligible[8,9].

The second interesting feature is the occurrence of two invariant points at ($\alpha_1 = 1$, $\alpha_2 = 2$ in Fig. 1) and ($\alpha_1 = 2$, $\alpha_2 = 1$ in Fig. 2). At these 'isoparametric' points $\alpha_1$ and $\alpha_2$ are 'independent' of $\mu_2/\mu_1$,
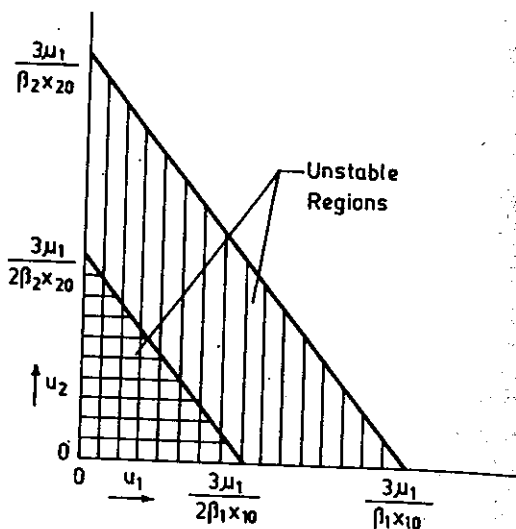


Fig. 3—Instability regions for the homogeneous steady state [Eq. (18)]

and this fact may be exploited to prescribe environmental conditions which will produce the desired concentration profiles, $u_1$ and $u_2$, of the two species.

Given a set ($\alpha_1$, $\alpha_2$), we can obtain regions of stability of the homogeneous state through Eqs (13) and (15). As an illustration, let $\alpha_1 = 1$, $\alpha_2 = 2$ for case (i) and $\alpha_1 = 2$, $\alpha_2 = 1$ for case (ii). Then as stated before, Eq. (13) implies Eq. (15) and can be written as

$$\frac{u_1 \beta_1 x_{10} + u_2 \beta_2 x_{20}}{\mu_1} > \begin{bmatrix} 3 \text{ [case (i)]} \\ 3/2 \text{ [case (ii)]} \end{bmatrix} \quad \ldots (18)$$

These two inequalities are shown graphically in Fig. 3. We observe that the areas of the unstable regions vary inversely with $x_{10}$ and $x_{20}$. This is plausible because low initial concentrations of the two species implies less crowding, less competition for resources and lower concentrations of inhibitory products, all of which promote stable coexistence.

## Diffusion-driven instability

For a homogeneous steady state to be destabilised by diffusion, resulting in pattern formation, inequalities (9) and (10) must also be satisfied. On reading condition (9) with condition (7), we deduce that

$$d < \frac{\partial f_2/\partial u_2}{\partial f_1/\partial u_1} < 1 \quad \ldots (19)$$

i.e., the diffusion coefficient of the second species should be smaller than that of the first.
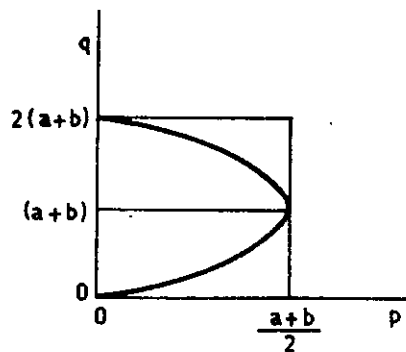
Condition (10) may be expanded as

Fig. 4—Bifurcation diagram of the threshold for Turing instability [Eq. (26)]

$$d^2 \left( \frac{\partial f_1}{\partial u_1} \right)^2 + \left( \frac{\partial f_2}{\partial u_2} \right)^2$$

$$+ 2d \left[ 2 \frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} \right] > 0$$

and this is ensured by

$$2 \frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} > \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} \qquad \ldots (20)$$

Eqs (20) and (8) may be combined to yield

$$\frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} < \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} < 2 \frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_1} \qquad \ldots (21)$$

The inequality on the left has been shown earlier to result in Eq. (15). Upon substituting the expressions for the partial derivatives, the second inequality can be written as:

$$(\beta_1 x_{10} u_1 + \beta_2 x_{20} u_2)^2 < \beta_1 \beta_2 x_{10} x_{20} u_1 u_2$$

$$+ (\mu_1/\alpha_1 + 2\mu_2/\alpha_2)\beta_1 x_{10} u_1$$

$$+ (2\mu_1/\alpha_1 + \mu_2/\alpha_2)\beta_2 x_{20} u_2 \qquad \ldots (22)$$

This is the condition for the onset of Turing instability in a homogeneous, linearly stable Lotka-Volterra environment.

If either of the pairs of values $(u_1, u_2)$ or $(x_{10}, x_{20})$ is very small, Eq. (22) may be approximated as

$$\left( \frac{\mu_1}{\alpha_1} + \frac{2\mu_2}{\alpha_2} \right) \beta_1 x_{10} u_1 + \left( \frac{2\mu_1}{\alpha_1} + \frac{\mu_2}{\alpha_2} \right) \beta_2 x_{20} u_2 > 0 \quad \ldots (23)$$

provided the other pair of values is not so large that either $u_1 x_{10}$ or $u_2 x_{20}$ becomes significantly greater than zero. Since inequality (23) is always

true, it is instructive to consider the physical implications of the conditions that lead to it. Recall that $u_1 = x_1/x_{10}$ and $u_2 = x_2/x_{20}$. The requirement of $u_1 x_{10} < 1$ and $u_2 x_{20} < 1$ may be met either by small inoculum concentrations or by conditions of low specific growth rates $\mu_1$ and $\mu_2$. In both situations $x_1$ and $x_2$ remain small. Thus, diffusion-driven instability is favoured in the early stages of growth and at low cell mass concentrations. Pattern formation is therefore less likely to be observed in rapidly growing species or in crowded populations, where one species tends to annihilate the other. May's[10] and Pianka's[8] analyses also support these deductions.

There are, however, limits to how small $\mu_1$ and $\mu_2$ may be allowed to be. If $\mu_1 < < 1$ and $\mu_2 < < 1$, Eq. (22) may be simplified to

$$(\beta_1 x_{10} u_1)^2 + (\beta_2 x_{20} u_2)^2 + \beta_1 x_{10} u_1 \beta_2 x_{20} u_2 < 0 \quad \ldots (24)$$

This is clearly not possible since all the terms are positive.

To analyse this further we return to the original instability condition, Eq. (22). Let $p = \beta_1 x_{10} u_1$ and $q = \beta_2 x_{20} u_2$. Then the threshold for Turing instability can be expressed as

$$p^2 - aq = bq - q^2 \qquad \ldots (25)$$

where

$a = \mu_1/\alpha_1 + 2\mu_2/\alpha_2$ and $b = 2\mu_1/\alpha_1 + \mu_2/\alpha_2$. Eq. (25) has two solutions

$$2q = (a + b) \pm \sqrt{(a + b)^2 - 4p^2} \qquad \ldots (26)$$

All real solutions for $q$ and $p$ lie on two branches in the interval $0 < p < (a + b)/2$. This produces the parabolic profile of Fig. 4.

Now, $a$ and $b$ are specified by the metabolic properties of the two microorganisms whereas $p$ and $q$ depend on the initial conditions and the time-varying concentrations. These latter two variables can therefore be sensitive to input fluctuations. It is known that such sensitivity can change the concentration patterns generated by the interaction of diffusion and reaction[11]. Since the system is at the threshold of instability, an induced change in $p$ or $q$ may cause the solution (Fig. 4) to 'jump' from one branch to another. In other words, even in an environment which is stable in the presence of diffusion, parametric sensitivity can trigger Turing instability and spontaneous pattern formation. This phenomenon has been observed in catalytic reactions[12].

## Conclusions

An analysis of homogeneous stability and diffusion-driven instability in a closed environment with two competing microorganisms shows that the ratio of the maximum specific growth rates has a strong effect on the stability regions. Competition has significance mainly when both species have comparable growth rates; otherwise the dynamics of the weaker species is practically unaffected by changes in the system parameters.

For diffusion to destabilise a homogeneous system, the two microorganisms must have unequal diffusivities. Turing instability is also favoured by low cell mass concentrations (such as through small inoculum size and low specific growth rates). The threshold of instability has a parabolic profile, and input fluctuations can trigger instability in an environment which is stable even in the presence of diffusion.

## Acknowledgement

## Nomenclature

$D_1$    = diffusivity of species 1, $m^2/s$
$D_2$    = diffusivity of species 2, $m^2/s$
$L$    = characteristic dimension, m
$t$    = time, s
$u_1$    = $x_1/x_{10}$, dimensionless
$u_2$    = $x_2/x_{20}$, dimensionless
$x_1$    = concentration of species 1, $kg/m^3$
$x_{10}$    = initial value of $x_1$, $kg/m^3$
$x_2$    = concentration of species 2, $kg/m^3$
$x_{20}$    = initial value of $x_2$, $kg/m^3$
$z$    = characteristic distance, m

*Greek letters*

$\alpha_1, \alpha_2$    = parameters in the Lotka-Volterra model, $1/s$
$\beta_1, \beta_2$    = parameters in the Lotka-Volterra model, $m^3/kg$
$\mu_1$    = maximum specific growth rate of species 1, $1/s$
$\mu_2$    = maximum specific growth rate of species 2, $1/s$

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